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Aggregate Estimates of Reflexive Collective Behavior Dynamics in a Cournot Oligopoly Model

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Abstract—This paper considers an oligopoly model with an arbitrary number of Cournot-reflexive agents under incomplete information in the classical case of linear cost and demand functions. Agents make decisions based on a collective behavior model. The problem of identifying convergence conditions to the model equilibrium is studied, with an emphasis placed on the trajectory of the sum of the action residuals of all agents. Aggregate estimates of this trajectory are obtained. They can be used to judge how the trajectory of each agent evolves towards the equilibrium.

Keywords: Cournot oligopoly, reflexive behavior, trajectory residuals, aggregate estimates, convergence conditions

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1. INTRODUCTION

In research areas dealing with behavioral models of rational agents under incomplete information, considerable attention is paid to the oligopoly market. Game theory was the first scientific method to analyze oligopoly [1]. Recent advances in oligopoly game theory were reviewed in [2].

Collective behavior theory models complement game-theoretic models by providing an opportunity to study the behavior dynamics of rational agents under rather weak assumptions about their awareness [3, 4]. The agent's dynamic decision process is based on reflexive thinking about his best choice considering the best responses of the other competitors. The dynamics are guided by the agents' choices. The determinative effect of reflexion is reaching an equilibrium [4, 5].

A significant number of studies of game-theoretic and collective behavior models in competitive markets are devoted to the problem of identifying conditions for the existence and uniqueness of an equilibrium as well as the convergence of agents' trajectories to it. For example, see [6–16] and many other related publications.

By accepting a certain hypothesis about the behavior of agents and their interaction, it is possible to calculate the trajectory of each agent (the first approach). However, an aggregate description of the behavior of the entire system, without characterizing the behavior of each agent in detail, seems more appropriate (the second approach) for several reasons. Obviously, the activity of individual agents at separate time instants cannot noticeably affect the convergence of trajectories. Besides, the growth of the number of agents in the market, their trajectories, and the computing time make the first approach less and less attractive. In some cases, the asymptotic convergence of the calculated trajectories can be judged only after a considerable time, particularly when the process evolves ambiguously or slowly, and the trend appears late. An intuitively aggregate description of the collective behavior of a system of agents on considerable time intervals can be no less accurate than a detailed one.

2. FORMAL PROBLEM STATEMENT

As the basic model, we consider the Cournot oligopoly of n agents [1] competing in the outputs of homogeneous products. By assumption, demand is determined by the function (the inverse demand function depending on the total output of agents)

$$p(Q) = a - bQ, \tag{1}$$

where $p(Q)$ is the uniform market price, $Q = \sum_{i \in N} q_i$ is the total output of all n agents; q_i is the output of agent i, $i \in N = \{1, \ldots, n\}$; a and b are parameters. The parameter a characterizes the maximum possible price of the product under which the demand will tend to zero; the parameter b characterizes the slope of the demand curve.

The total costs of agents have the form

$$\phi_i(q_i) = c_i q_i + d_i, \tag{2}$$

where c_i and d_i are the marginal and fixed costs of agent i, respectively. The goal functions of agents are given by

$$\Pi_i(p(Q), q_i) = p(Q)q_i - \phi_i(q_i) \to \max_{q_i}. \tag{3}$$

By assumption, everything produced is sold, no capacity constraints are imposed, and no coalitions are allowed. The market state at a time instant t ($t = 0, 1, 2, \ldots$) is described by the n-dimensional vector $q^t = (q_1^t, \ldots, q_i^t, \ldots, q_n^t)$.

Agents are said to be collectively competitive in a Cournot market if each agent i satisfies the marginal costs constraint

$$c_i < \frac{a + \sum_{j \in N \setminus \{i\}} c_j}{n}. \tag{4}$$

In this case, each agent is supposed to be competitive and, as in a normal form game, there exists a unique (static Nash) equilibrium [17] $q^* = (q_1^*, \ldots, q_i^*, \ldots, q_n^*)$ with $q_i^* > 0$ $\forall i \in N$ in the Cournot oligopoly model; for example, see [18].

Under game uncertainty (about the actions chosen by the competitive environment) and incomplete knowledge (of costs, goal functions, and other attributes of competitors), market equilibrium can usually be reached not by one-time decision-making by agents but as the result of an iterative reflexive process [3, 4, 18–20].

Consider a reflexive discrete process where the change of market states satisfies the axiom of indicator behavior [4] as follows: at each time instant $(t + 1)$, each agent observes the outputs of all agents chosen at the previous time instant t and adjusts his output by taking a step towards the current position of his goal, $x_i(q_{-i}^t)$, in the iterative procedure

$$q_i^{t+1} = q_i^t + \gamma_i^{t+1}(x_i(q_{-i}^t) - q_i^t), \quad i \in N. \tag{5}$$

Here, the parameter $\gamma_i^{t+1} \in [0, 1]$, independently chosen by each agent i, determines his step towards the current position of his goal. An agent can take a full step with $\gamma_i^{t+1} = 1$ (his best response), remain on the spot with $\gamma_i^{t+1} = 0$, or take a partial step with $\gamma_i^{t+1} \in (0, 1)$.

For agent i, the current position $x_i(q_{-i}^t)$ of his goal is an output maximizing his goal function provided that at the current time instant, the other agents choose the same output as at the previous time instant [4, 21]. Here, $q_{-i}^t = (q_1^t, \ldots, q_{i-1}^t, q_{i+1}^t, \ldots, q_n^t)$ is the opponents' output profile for agent i (the output vector of all agents except agent i) at the time instant t. According to [10, 18]),

$$x_i(q_{-i}^t) = \frac{h_i - \sum_{j \neq i} q_j^t}{2} = \frac{h_i - Q_{-i}^t}{2}, \tag{6}$$

where $h_i = \frac{a-c_i}{b}$ is the volume of a perfectly competitive market under the marginal cost pricing $p(Q) = c_i$ (the perfectly competitive output of firm i); $Q^t_{-i} = \sum_{j \neq i} q^t_j$ is the total output of the environment of agent i $(i, j \in N)$.

Let us derive the variable $x_i(q^t_{-i})$ to show its relation to q^t_i. Using (1)–(3) for time instant t, we obtain

$$\frac{\partial \prod_i^t}{\partial q_i^t} = a - bq_i^t - bQ_{-i}^t + \left(-b - b\frac{\partial Q_{-i}^t}{\partial q_i^t}\right) q_i^t - c_i = 0$$

and

$$q_i^t = \frac{(a - c_i)/b - Q_{-i}^t}{2 + \partial Q_{-i}^t / \partial q_i^t}.$$

Under the Cournot assumption, the output of the agent's environment output will not change if he changes his output. Therefore, $\frac{\partial Q^t_{-i}}{\partial q^t_i} = 0$ and the agent's optimal output q^t_i is $\frac{(a-c_i)/b - \sum_{j \in N \setminus \{i\}} q^t_j}{2}$. This optimal output will be the current position $x_i(q^t_{-i})$ of the agent's goal in (6).

For model (5)–(6), the trajectory of agent i is understood as the sequence of outputs $q^0_i, q^1_i, \ldots,$ $q^t_i, \ldots$ *realized* within this model.

The problem of identifying convergence conditions of agents' trajectories (5)–(6) to equilibrium and its modifications in the classical Cournot oligopoly model (1)–(4) were considered by many researchers; for example, see [18–20, 22–26] and other publications on the subject.

The peculiarity and novelty of the study presented below are the development of an aggregate description of the entire system's behavior to judge how the trajectory of each agent evolves towards the equilibrium.

In this study, we sequentially solve the following tasks:

1) reduce agents' models to a homogeneous form;

2) investigate the effect of total residuals on the convergence of agents' trajectories;

3) describe in aggregate terms the transformation (recalculation) of the total residuals during the transition between time instants;

4) describe in aggregate terms (estimate) the dynamics of the total residuals over the set of time instants;

5) form convergence conditions of agents' trajectories to the equilibrium using the aggregate estimates of collective behavior model dynamics.

As intuition suggests, aggregates are preferably formed from homogeneous elements. Ideally, homogeneous agents may differ only by their choice of the parameter γ. Let us analyze the possibility of the ideal case for the market model (1)–(4) and the iterative process (5)–(6).

The next section begins with this task.

3. THE METHODS AND RESULTS OF THIS STUDY

Assume that within model (1)–(4), the process (5)–(6) converges under step parameters $\{\gamma^{t+1}_i\}$ $(i \in N; t = 0, 1, 2, \ldots)$ and the marginal costs $c = (c_1, \ldots, c_i, \ldots c_n)$ of the agents.

Will this process converge under the same step parameters if the marginal costs of agents or market parameters change?

To answer this question, we introduce the change of variables

$$\varepsilon^t_i = q^*_i - q^t_i \quad (i \in N; t = 0, 1, 2, \ldots), \tag{7}$$

where q^*_i and q^t_i are the equilibrium and current outputs of agent i, respectively.

Using the well-known Cournot equilibrium relations $h_i = Q^* + q_i^*$, we transform (5)–(6) as follows:

$$q_i^* - q_i^{t+1} = q_i^* - q_i^t - \gamma_i^{t+1}\left(\frac{h_i - Q_{-i}^t}{2} - q_i^t\right) = q_i^* - q_i^t - \gamma_i^{t+1}\left(\frac{Q^* + q_i^* - Q_{-i}^t}{2} - q_i^t\right)$$
$$= q_i^* - q_i^t - \gamma_i^{t+1}\left(\frac{\sum_{j\neq i}(q_j^* - q_j^t) + 2q_i^*}{2} - q_i^t\right).$$

As a result,

$$\varepsilon_i^{t+1} = \varepsilon_i^t + \gamma_i^{t+1}\left(-\frac{\sum_{j\neq i}\varepsilon_j^t}{2} - \varepsilon_i^t\right). \tag{8}$$

Here, by analogy with (1)–(6), $\left(-\frac{\sum_{j\neq i}\varepsilon_j^t}{2}\right)$ is the current position of the goal of agent i, and (8) is the indicator behavior model, and the agent's goal function has the form

$$\Pi_i(\varepsilon) = -\left(\sum_{j\in N}\varepsilon_j\right)\varepsilon_i \to \max_{\varepsilon_i}, \tag{9}$$

where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_i, \ldots, \varepsilon_n)$.

Let us demonstrate this fact. For time instant t, we find the optimal residuals of agent i using (9):

$$\frac{\partial \prod_i^t}{\partial \varepsilon_i^t} = -\varepsilon_i^t - \sum_{j\in N\setminus\{i\}}\varepsilon_j^t - \left(1 + \frac{\partial \sum_{j\in N\setminus\{i\}}\varepsilon_j^t}{\partial \varepsilon_i^t}\right)\varepsilon_i^t.$$

Consequently,

$$\varepsilon_i^t = -\sum_{j\in N\setminus\{i\}}\varepsilon_j^t \Big/ \left(2 + \frac{\partial \sum_{j\in N\setminus\{i\}}\varepsilon_j^t}{\partial \varepsilon_i^t}\right).$$

By the Cournot assumption, the agent's environment will not change its output if he does so. Obviously, this assumption also applies to the residuals. Therefore, $\frac{\partial \sum_{j\in N\setminus\{i\}}\varepsilon_j^t}{\partial \varepsilon_i^t} = 0$, and the current position of his goal (the optimal residuals at the current time instant) is $\left(-\frac{\sum_{j\in N\setminus\{i\}}\varepsilon_j^t}{2}\right)$.

For model (8), the trajectory of agent i will be understood as the sequence of residuals $\varepsilon_i^0, \varepsilon_i^1, \ldots, \varepsilon_i^t, \ldots$ realized within this model.

The convergence of (8) means that $\varepsilon_i^t \to \varepsilon_i^* = 0$ as $t \to \infty$.

Proposition 1. *The indicator behavior process* (8) *converges if and only if the process* (5)–(6) *converges within the model* (1)–(4).

The proof of this proposition is provided in the Appendix.

The next proposition shows that the convergence of the trajectory of the total residuals of all agents is sufficient for the convergence of the trajectory of each agent.

Proposition 2. *If* $\sum_{j\in N}\varepsilon_j^t \to 0$ *as* $t \to \infty$, *then* $\varepsilon_j^t \to \varepsilon_j^* = 0\ \forall j \in N$.

This fact is proved in the Appendix using mathematical induction.

Note. The equality $\sum_{j\in N}\varepsilon_j^t = 0$ for (8) implies the equalities $|\varepsilon_j^{t+1}| = (1 - \gamma_j^{t+1}/2)|\varepsilon_j^t|$ and the inequalities $|\varepsilon_j^{t+1}| < |\varepsilon_j^t|$ under $\gamma_j^{t+1} \neq 0\ \forall j \in N$. They indicate that, in the case under consideration, the trajectories of all agents at time instant $(t+1)$ will be closer to the equilibrium than at time instant t.

The final result for model (1)–(6), which essentially follows from Proposition 1, is presented below.

Theorem 1. *In the linear Cournot model* (1)–(4) *with competitive agents, the market parameters a and b and the agents' parameters c_i and d_i do not affect the steps $\{\gamma_i^t\}_{t=1,2,\ldots}$ ensuring the convergence of the indicator behavior model* (5)–(6).

Let us mention the advantages of studying the equilibrium problem based on model (8):

– The model is indifferent to the market and agents' cost parameters (a, b, c_i, d_i), which simplifies the analysis procedure.

– The agents and their trajectories differ merely in the choice of the parameter γ and initial data. Only the first factor matters here, since we are interested in convergence conditions for any initial data.

– In model (5)–(6), economic constraints may require that the agents' current outputs be nonnegative. In model (8), there is no basis for such nonnegativity requirements.

Now we proceed to the next problem, which is important for the convergence of the trajectory of each agent (see Proposition 2). Let us discuss conditions on the parameters γ under which $\sum_{j\in N}\varepsilon_j^t \to 0$.

Within model (8), we have $\sum\limits_{j\in N}\varepsilon_j^{t+1} = \left(1 - \sum\limits_{j\in N}\gamma_j^{t+1}/2\right)\sum\limits_{j\in N}\varepsilon_j^{t+1} - \sum\limits_{j\in N}\varepsilon_j^t\gamma_j^{t+1}/2$.

For $\gamma_j^{t+1} \equiv 1 \;\; \forall j \in N$, it follows that $\sum\limits_{j\in N}\varepsilon_j^{t+1} = (1-(1+n)/2)\sum\limits_{j\in N}\varepsilon_j^{t+1}$.

For $\gamma_j^{t+1} \equiv 0 \;\; \forall j \in N$, the result is $\sum\limits_{j\in N}\varepsilon_j^{t+1} = \sum\limits_{j\in N}\varepsilon_j^t$.

Thus, if $\sum_{j\in N}\varepsilon_j^t \neq 0$, then there exists a value of the parameter $\tilde{\gamma}^{t+1}$ such that

$$\sum_{j\in N}\varepsilon_j^{t+1} = (1-\tilde{\gamma}^{t+1}(1+n)/2)\sum_{j\in N}\varepsilon_j^t. \tag{10}$$

The value of the parameter $\tilde{\gamma}^{t+1}$ is given by

$$\tilde{\gamma}^{t+1} = \left(\sum_{j\in N}\gamma_j^{t+1} + \sum_{j\in N}\gamma_j^{t+1}\varepsilon_j^t / \sum_{j\in N}\varepsilon_j^t\right) / (1+n)\,, \tag{11}$$

where $\tilde{\gamma}^{t+1}$ *is the arithmetic weighted mean* of the set of the parameters $\{\gamma_j^{t+1}\}_{j\in N}$ with weights $\{\omega_j^t\}_{j\in N}$, i.e., $\tilde{\gamma}^{t+1} = \frac{\sum_{j\in N}\omega_j^t\gamma_j^{t+1}}{\sum_{j\in N}\omega_j^t}$ and the weights are the real numbers $\omega_i^t = \sum_{j\in N}\varepsilon_j^t + \varepsilon_i^t$.

The negative contribution of an agent to the parameter $\tilde{\gamma}^{t+1}$ is possible if the sign of his residuals does not coincide with that of the total residuals $\sum_{j\in N}\varepsilon_j^t = Q^* - Q^t$ of all agents.

Let some time instants t_0 and τ, $\tau > t_0$, be fixed. Assuming that $\sum_{j\in N}\varepsilon_j^{t_0} \neq 0$ and $\tilde{\gamma}^t \neq \frac{2}{1+n}$, for $t_0 + 1 \leqslant t \leqslant \tau$ we sequentially obtain

$$\sum_{j\in N}\varepsilon_j^{t_0+\tau} = \sum_{j\in N}\varepsilon_j^{t_0}\prod_{t=t_0+1}^{\tau}(1-\tilde{\gamma}^t(1+n)/2) \tag{12}$$

based on (10). (The cases in which $\tilde{\gamma}^t = \frac{2}{1+n}$ and, accordingly, $\sum_{j\in N}\varepsilon_j^t = 0$, have been discussed in the note to Proposition 2.)

Let us introduce the notations

$$T = \{t_0+1,\ldots,\tau\}, \quad T_+ = \{t \in T | 1-\tilde{\gamma}^t(1+n)/2 > 0\},$$
$$T_- = \{t \in T | 1-\tilde{\gamma}^t(1+n)/2 < 0\}.$$

Two cases of possible inequalities can occur at some time instants from set $T_+ : 1 \leqslant 1-\tilde{\gamma}^t(1+n)/2$ (if $\tilde{\gamma}^t \leqslant 0$) and $0 < \tilde{\gamma}^t(1+n)/2 < 1$ (if $0 < \tilde{\gamma}^t < \frac{2}{1+n}$). The first case is "unfavorable" and the second "favorable" for the convergence of the process. Similarly, in the set T_-, there also exist favorable time instants when $-1 < 1-\tilde{\gamma}^t(1+n)/2 < 0$ (if $\frac{2}{1+n} < \tilde{\gamma}^t < \frac{4}{1+n}$), and unfavorable ones when $1-\tilde{\gamma}^t(1+n)/2 < -1$ (if $\frac{4}{1+n} \leqslant \tilde{\gamma}^t$).

Using the Cauchy inequality between the arithmetic mean and the geometric mean, we have

$$\prod_{t\in T_+}(1-\tilde{\gamma}^t(1+n)/2) \leqslant \left[\frac{1}{t_+}\sum_{t\in T_+}(1-\tilde{\gamma}^t(1+n)/2)\right]^{t_+} = \left[1-\bar{\tilde{\gamma}}_+^\tau(1+n)/2\right]^{t_+},$$
$$\prod_{t\in T_-}(\tilde{\gamma}^t(1+n)/2-1) \leqslant \left[\frac{1}{t_-}\sum_{t\in T_-}(\tilde{\gamma}^t(1+n)/2-1)\right]^{t_-} = \left[\bar{\tilde{\gamma}}_-^\tau(1+n)/2-1\right]^{t_-},$$

where $\bar{\tilde{\gamma}}_+^\tau = \frac{1}{t_+}\sum_{t\in T_+}\tilde{\gamma}^t$ and $\bar{\tilde{\gamma}}_-^\tau = \frac{1}{t_-}\sum_{t\in T_-}\tilde{\gamma}^t$ are the mean values of the weighted parameter $\tilde{\gamma}^t$ over the sets T_+ and T_-, respectively; $t_+(t_-)$ is the number of time instants in the set T_+ (T_-, respectively), and $\tau = t_+ + t_-$.

In view of (12), it follows that

$$\left|\sum_{j\in N}\varepsilon_j^{t_0+\tau}\right| \leqslant \left[1-\bar{\tilde{\gamma}}_+^\tau(1+n)/2\right]^{t_+}\left|1-\bar{\tilde{\gamma}}_-^\tau(1+n)/2\right|^{t_-}\left|\sum_{j\in N}\varepsilon_j^{t_0}\right|. \tag{13}$$

Inequality (13) and Proposition 2 lead to the following result regarding the convergence of the process (8).

Proposition 3. *Model* (8) *converges to the equilibrium if, for an arbitrary* t_0 *and* $\tau > t_0$, *the expression* $\left[1-\bar{\tilde{\gamma}}_+^\tau(1+n)/2\right]^{t_+}\left|1-\bar{\tilde{\gamma}}_-^\tau(1+n)/2\right|^{t_-}$ $(\tau = t_+ + t_-)$ *vanishes as* $\tau \to \infty$.

Corollary 1. *If* $\exists t_0$ *such that* $0 < \bar{\tilde{\gamma}}_+^\tau$ *and* $\bar{\tilde{\gamma}}_-^\tau < \frac{4}{n+1}$ $\forall \tau > t_0$, *then model* (8) *converges to the equilibrium.*

Let us explain the validity of this corollary. If $t \in T_+$, then $\tilde{\gamma}^t < \frac{2}{1+n}$ and $\bar{\tilde{\gamma}}_+^\tau < \frac{2}{1+n}$. If $t \in T_-$, then $\tilde{\gamma}^t > \frac{2}{1+n}$ and, consequently, $\bar{\tilde{\gamma}}_-^\tau > \frac{2}{1+n}$. Therefore, for the inequalities $\left[1-\bar{\tilde{\gamma}}_+^\tau(1+n)/2\right] < 1$ and $\left|1-\bar{\tilde{\gamma}}_-^\tau(1+n)/2\right| < 1$ to be true, it suffices to require $0 < \bar{\tilde{\gamma}}_+^\tau$ and $\bar{\tilde{\gamma}}_-^\tau < \frac{4}{1+n}$.

In addition, by Propositions 1 and 3, model (5)–(6) converges for the sets of parameters $\{\gamma_i^t\}$ whose averaged estimates satisfy the inequalities $0 < \bar{\tilde{\gamma}}_+^\tau$ and $\bar{\tilde{\gamma}}_-^\tau < \frac{4}{1+n}$.

Corollary 2. *Let the parameters* $\tilde{\gamma}^t$, $t = (1, 2, \ldots, \tau)$, *be random variables, and let* $\bar{\tilde{\gamma}}^\tau = \frac{1}{\tau}\sum_{t=1}^{\tau}\tilde{\gamma}^t$ *converge in probability to the overall mean* $\bar{\tilde{\gamma}}$. *Then model* (8) *converges in probability to the equilibrium as* $\tau \to \infty$ *if* $\bar{\tilde{\gamma}} < \frac{4}{1+n}$.

Numerical calculation: a brief example. Consider model (1)–(3) with $n = 4$, $a = 100$, $b = 0.1$, $c = (20; 25; 20; 30)$, and $q^0 = (250; 250; 100; 200)$. All agents bear the same fixed costs of 500. By the formula $h_i = \frac{a-c_i}{b}$ we find $h = (800; 750; 800; 700)$.

When the agents are completely informed, the static Nash equilibrium q^* is the solution of the system of linear algebraic equations $q_i + Q = h_i$ $(i = \overline{1,4})$. As a result, $q_i^* = h_i - \frac{1}{5}\sum_{j=1}^{4}h_j$ and $q^* = (190; 140; 190; 90)$.

One fragment of the dynamics for $n = 4$

Time instants	Residuals of agents' actions				Step parameters							
t	ε_1	ε_2	ε_3	ε_4	γ_1	γ_2	γ_3	γ_4	$\sum \varepsilon_j$	$\tilde{\gamma}$	$\bar{\bar{\gamma}}_+$	$\bar{\bar{\gamma}}_-$
1	2	3	4	5	6	7	8	9	10	11	12	13
0	−60.0	−110.0	90.0	−110.0					−190.0			
1	−10.0	−65.0	105.0	−72.5	0.40	0.30	0.30	0.25	−42.5	0.31	0.31	
2	0.5	−48.9	95.6	−61.0	0.40	0.30	0.30	0.20	−13.8	0.27	0.29	
3	3.2	−39.5	62.9	−51.7	0.40	0.30	0.80	0.25	−25.1	−0.33	0.08	
4	7.5	−29.8	57.2	−42.1	0.40	0.30	0.30	0.25	−7.1	0.29	0.13	
5	7.5	−22.4	39.7	−35.9	0.40	0.40	0.70	0.25	−11.2	−0.23	0.06	
6	8.2	−15.7	35.4	−30.0	0.40	0.40	0.30	0.25	−2.1	0.32	0.11	
7	6.1	−13.0	30.4	−26.0	0.70	0.30	0.30	0.25	−2.5	−0.08	0.08	
8	5.4	−9.9	26.2	−22.4	0.40	0.40	0.30	0.25	−0.7	0.28	0.10	
9	5.1	−8.3	22.4	−19.5	0.10	0.30	0.30	0.25	−0.3	0.24	0.12	
10	4.7	−7.0	19.1	−17.1	0.20	0.30	0.30	0.25	−0.3	−0.03	0.10	
11	3.4	−3.3	14.4	−10.1	0.60	1.00	0.50	0.80	4.3	5.61		5.61
12	−0.5	−3.8	5.0	−7.2	1.00	1.00	1.00	1.00	−6.5	1.00		3.31
13	3.0	1.3	5.4	−0.4	1.00	1.00	0.50	1.00	9.3	0.98		2.53
14	−3.2	−4.0	−2.0	−4.9	1.00	1.00	1.00	1.00	−14.0	1.00		2.15
15	5.4	5.0	6.0	4.6	1.00	1.00	1.00	1.00	21.0	1.00		1.92
16	−7.8	−8.0	−7.5	−8.2	1.00	1.00	1.00	1.00	−31.5	1.00		1.77
17	11.9	11.8	12.0	11.7	1.00	1.00	1.00	1.00	47.3	1.00		1.66
18	−17.7	−17.8	−17.6	−17.8	1.00	1.00	1.00	1.00	−71.0	1.00		1.57
19	26.6	26.6	26.7	26.6	1.00	1.00	1.00	1.00	106.5	1.00		1.51
20	−39.9	−6.7	−26.6	−39.9	1.00	0.50	0.80	1.00	−113.1	0.82		1.44
21	36.6	23.3	8.3	36.6	1.00	0.50	0.50	1.00	104.8	0.77		0.77
22	−34.1	−8.7	−19.9	−34.1	1.00	0.50	0.50	1.00	−96.9	0.77		0.77
23	31.4	17.7	9.3	31.4	1.00	0.50	0.50	1.00	89.7	0.77		0.77
24	−11.0	−9.2	−15.5	−29.2	0.70	0.50	0.50	1.00	−64.8	0.69		0.75
25	26.9	9.3	4.6	−5.7	1.00	0.50	0.50	0.50	35.2	0.62		0.72
26	−4.1	−1.8	−5.3	−20.4	1.00	0.50	0.50	1.00	−31.7	0.76		0.73
27	13.8	6.6	3.9	5.6	1.00	0.50	0.50	1.00	29.9	0.78		0.74
28	5.0	−2.5	−4.5	−12.1	0.40	0.50	0.50	1.00	−14.2	0.59		0.72
29	9.6	1.6	0.1	1.0	1.00	0.50	0.50	1.00	12.4	0.75		0.72
30	−1.4	−1.9	−3.0	−5.7	1.00	0.50	0.50	1.00	−12.0	0.79		0.73

Note: the notations in the table header are provided without the superscript "t," which is supposed to be the corresponding time instant (period or iteration) from column 1.

The transition to model (8) is performed using the formula for calculating the residuals $\varepsilon_i^t = q_i^* - q_i^t$. We have $\varepsilon^0 = (-60; -110; 90; -110)$.

For each time instant (column 1), the current values of the residuals ε_i^t (columns 2–5), and the current values of the parameters γ_i^t (columns 6–9), the table presents the *arithmetic weighted means* $\tilde{\gamma}^t$ of the sets of parameters $\{\gamma_j^t\}$ (column 11) calculated using formulas (8) and (11).

The convergence of the process depends on what steps γ the agents choose. As is known, if the agents take the maximum (unit) steps γ, the process converges only in the case $n = 2$. For $n = 4$, the process diverges if all agents take steps exceeding 0.8. Generally speaking, for $n = 4$ the question of convergence remains open if the agents act differently: some agents choose steps greater than 0.8 while the others smaller.

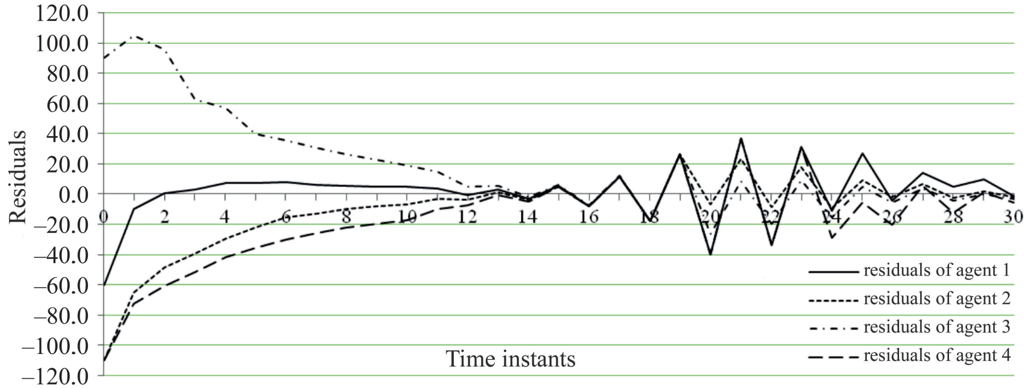

Fig. 1. The dynamics of action residuals of individual agents.

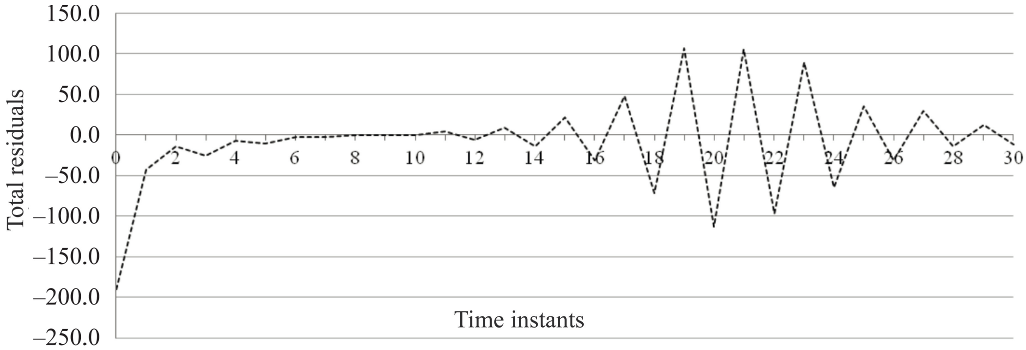

Fig. 2. The dynamics of total residuals.

To visualize the conditions introduced above and indicate the convergence of the process, we separate 3 intervals containing 10 time instants each. During the first 10 time instants, the agents intentionally take small steps to ensure convergence. During the next 10 time instants, larger steps are selected to show the divergence of the process. During the final 10 time instants, the process is again made convergent due to small steps taken by some agents.

Consider the process dynamics in more detail. At each of the first 10 time instants, the weighted means $\tilde{\gamma}^t$ do not exceed $2/(1+n) = 0.4$ (there are two negative values), and the aggregate estimate $\bar{\tilde{\gamma}}_+^\tau$ (their *arithmetic mean* over t time instants, see column 12) lies in the range $(0, 0.4)$. Therefore, we observe a trend for decreasing the absolute value of the total residuals: from $\varepsilon^0 = (-60; -110; 90; -110)$ and $\left|\sum_{j=1}^4 \varepsilon_j^0\right| = 190$ to $\varepsilon^{10} = (4.7; -7.0; 19.1; -17.1)$ and $\left|\sum_{j=1}^4 \varepsilon_j^{10}\right| = 0.3$.

During the next 10 time instants (from the 11th to the 20th), $\tilde{\gamma}^t$ and the aggregate dynamics estimate $\bar{\tilde{\gamma}}_-^\tau$ (column 13) exceed $4/(1+n) = 0.8$, which determines a trend for increasing the absolute value of the total residuals to $\varepsilon^{20} = (-39.9; -6.7; -26.6; -39.9)$ and $\left|\sum_{j=1}^4 \varepsilon_j^{20}\right| = 113.1$.

From the 21st to the 30th time instants, the *arithmetic weighted means* $\tilde{\gamma}^t$ and their *arithmetic mean* $\bar{\tilde{\gamma}}_-^\tau$ lie in the range $(0.4, 0.8)$. In other words, the convergence conditions of Proposition 3 and Corollary 1 are satisfied for $t_0 = 21$, and we observe a trend for decreasing the total residuals: $\varepsilon^{30} = (-1.4; -1.9; -3.0; -5.7)$ and $\left|\sum_{j=1}^4 \varepsilon_j^{30}\right| = 12.0$.

Also, we have $\bar{\tilde{\gamma}}^{30} = \frac{1}{30}\sum_{t=1}^{30} \tilde{\gamma}^t = 0.76$. According to the table data, the dynamics evolve towards the static Nash equilibrium.

Figures 1 and 2 demonstrate the "synchronization" of the dynamics of individual and total costs. Obviously, if the residuals of each agent converge, the total residuals of all agents will converge as well. The converse also follows from Proposition 2: if the total residuals vanish, the residuals of each agents will vanish too.

4. CONCLUSIONS

This paper has considered the problem of identifying convergence conditions of collective behavior models to equilibrium. The peculiarity and novelty of the approach presented above consist in the following. Traditionally, such conditions are formulated as admissible ranges for the values of the parameters γ for each agent at each time instant: if each agent chooses his parameter within such a range at each time instant, the dynamics surely converge. In this study, the convergence conditions have been formulated for a set of time instants (i.e., over a time interval) as an admissible range for the *arithmetic mean of the parameters*, which are the *arithmetic weighted means* of the set $\{\gamma_j^t\}_{j\in N}$ for each time instant t included in the time interval. If the mean over the time interval belongs to the range, the dynamics of each agent will evolve towards the equilibrium. In addition, if the set of time instants is unbounded, the dynamics of each agent will surely converge to the equilibrium.

APPENDIX

Proof of Proposition 1. First, we note the existence of a unique static Nash equilibrium in model (9) and $\varepsilon_i^* = 0 \quad \forall i \in N$.

The validity of Proposition 1 follows from the fact that one process is obtained from the other via a transformation based on the residuals as the change of variables. Therefore, the convergence of the process (5)–(6) directly implies the convergence of the process (8).

Now we employ mathematical induction to show that the convergence of the process (5)–(6) follows from the convergence of the process (8).

Let ε^0 be the initial data vector of the process (8) under which it converges. We define by $q^0 = q^* - \varepsilon^0$ the initial data vector of the process (5)–(6) and calculate ε^1 using formula (8). In view of $h_i = Q^* + q_i^*$ and (6), we establish that $q^1 = q^* - \varepsilon^1$:

$$q_i^* - \varepsilon_i^1 = q_i^* - \varepsilon_i^0 - \gamma_i^1\left(-\frac{\sum_{j\neq i}\varepsilon_j^0}{2} - \varepsilon_i^0\right) = q_i^0 - \gamma_i^1\left(-\frac{\sum_{j\neq i}(q_j^* - q_j^0)}{2} - (q_i^* - q_i^0)\right)$$
$$= q_i^0 + \gamma_i^1\left(\frac{2q_i^* + \sum_{j\neq i}(q_j^* - q_j^0)}{2} - q_i^0\right) = q_i^0 + \gamma_i^1\left(\frac{Q^* + q_i^* - \sum_{j\neq i} q_j^0}{2} - q_i^0\right)$$
$$= q_i^0 + \gamma_i^1\left(\frac{h_i - \sum_{j\neq i} q_j^0}{2} - q_i^0\right) = q_i^0 + \gamma_i^1(x_i(q_{-i}^0) - q_i^0) = q_i^1.$$

By analogy, it can be shown that $q^t = q^* - \varepsilon^t$ implies $q^{t+1} = q^* - \varepsilon^{t+1}$ for ε^{t+1} calculated using formula (8). Therefore, if $\varepsilon^t \to 0$, then $q^t \to q^*$.

The proof of Proposition 1 is complete.

Proof of Proposition 2. Suppose that $\sum_{j\in N}\varepsilon_j^t \to 0$. Then $\forall\delta > 0\ \exists t'$ such that $\left|\sum_{j\in N}\varepsilon_j^t\right| < \delta$ for $t > t'$.

Letting $t = t' + 1$ and using (8) yield

$$\left|\varepsilon_i^{t+1}\right| \leqslant (1 - \gamma_i^{t+1}/2)\left|\varepsilon_i^t\right| + \gamma_i^{t+1}/2\left|\sum_{j\in N}\varepsilon_j^t\right| < (1 - \gamma_i^{t+1}/2)\left|\varepsilon_i^t\right| + \delta\gamma_i^{t+1}/2 < (1 - \gamma_i^{t+1}/2)\left|\varepsilon_i^t\right| + \delta.$$

Based on (8) and the previous inequality, we have

$$\left|\varepsilon_i^{t+2}\right| \leqslant (1-\gamma_i^{t+2}/2)\left|\varepsilon_i^{t+1}\right| + \gamma_i^{t+2}/2\left|\sum_{j\in N}\varepsilon_j^{t+1}\right| < (1-\gamma_i^{t+2}/2)\left|\varepsilon_i^{t+1}\right| + \delta\gamma_i^{t+2}/2$$

$$< (1-\gamma_i^{t+2}/2)(1-\gamma_i^{t+1}/2)\left|\varepsilon_i^{t}\right| + \left[(1-\gamma_i^{t+2}/2)+\gamma_i^{t+2}/2\right]\delta$$

$$< (1-\gamma_i^{t+2}/2)(1-\gamma_i^{t+1}/2)\left|\varepsilon_i^{t}\right| + \delta.$$

Following the method of mathematical induction, we assume that at some time instant $(t+m)$,

$$\left|\varepsilon_i^{t+m}\right| < \left|\varepsilon_i^{t}\right| \times \prod_{l=1}^{m}(1-\gamma_i^{t+l}/2) + \delta.$$

Based on (8) and the last inequality,

$$\left|\varepsilon_i^{t+m+1}\right| \leqslant (1-\gamma_i^{t+m+1}/2)\left|\varepsilon_i^{t+m}\right| + \gamma_i^{t+m+1}/2\left|\sum_{j\in N}\varepsilon_j^{t+m}\right| < (1-\gamma_i^{t+m+1}/2)\left|\varepsilon_i^{t+m}\right| + \delta\gamma_i^{t+m+1}/2$$

$$< (1-\gamma_i^{t+m+1}/2)\left[\left|\varepsilon_i^{t}\right| \times \prod_{l=1}^{m}(1-\gamma_i^{t+l}/2)+\delta\right] + \delta\gamma_i^{t+m+1} = \left|\varepsilon_i^{t}\right| \times \prod_{l=1}^{m+1}(1-\gamma_i^{t+l}/2) + \delta.$$

In other words, the same inequality holds at time instant $(t+m+1)$.

In the inequalities derived for each time instant, the first term vanishes as $m \to \infty$ under $\gamma \neq 0$ and an arbitrarily small number δ. Therefore, the conclusion follows, and the proof of Proposition 2 is complete.

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